PHYSICAL REVIEW E VOLUME 53, NUMBER 4 APRIL 1996

First-passage-time statistics for diffusion processes with an external random force

Josep M. Porrà, Armando Robinson, and Jaume Masoliver
Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain
(Received 24 July 1995)

We present exact equations and expressions for the first-passage-time statistics of dynamical systems that are a combination of a diffusion process and a random external force modeled as dichotomous Markov noise. We prove that the mean first passage time for this system does not show any resonantlike behavior.

PACS number(s): 05.40+j

A considerable effort has been made in recent years to understand the many effects that can be generated by the combination of random and periodic signals acting on a dynamical system. One large area of current interest is that of stochastic resonance [1] in which a periodic driving field at an appropriate frequency can have a dramatic effect, either enhancing or depressing some system property otherwise mainly determined by noise. Most stochastic resonance phenomena are studied in the context of nonlinear systems, but there is also the so-called "coherent stochastic resonance," which are resonant effects that can appear in linear systems. These are exemplified by the significant decrease in the mean first passage time (MFPT) of a free particle out of an interval terminated by two traps, where the particle is simultaneously driven by an additive Brownian force and a periodic field [2–4]. Moreover, a similar resonantlike phenomenon has been obtained in the problem of a thermally activated potential barrier crossing in the presence of barrier fluctuations [5–7].

In this paper we want to address a different, although in some way related, problem of the first-passage-time statistics of a particle driven by an additive combination of a Brownian force and a fluctuation force. The Brownian force is the result of the influence of a heat bath while the fluctuation force accounts for a random external influence on the particle. Problems similar to this have been treated recently in the literature for first passage times of non-Markovian processes [8], for nonequilibrium fluctuation-induced transport [9], for Brownian motion with superimposed shot noise [10], for the growth of populations subject to catastrophic changes [11], and for the thermally activated process over fluctuating barriers [5–7]. As a model for the random external force we choose dichotomous Markov noise (sometimes referred to as "the random telegraph signal") since, on the one hand, this is the kind of applied fluctuation force used in some models for nonequilibrium transport [9] and, on the other hand, the random telegraph signal can be seen as a "random periodic field" that has the form of a square wave with a random frequency. This system is very similar to the models used in Refs. [3–7], which show resonantlike behavior. In this context it is interesting to analyze whether one can find some kind of resonance behavior due to the combination of signals.

The class of problems we will consider here concerns the evolution of the state variable $X(t)$ of a particle moving in a potential $V(x)$ under the influence of a heat bath at temperature T and an external random signal $F(t)$. In the overdamped regime the process $X(t)$ then obeys the dynamical equation

$$\dot{X}=f(X)+F(t)+\xi(t), \tag{1}$$

where $f(X)=-V'(X)$ is a deterministic force and $F(t)$ is a dichotomous Markov noise taking on values $\pm a$. The random time intervals between switches of $F(t)$ are governed by an exponential density

$$\psi(t)=\lambda e^{-\lambda t}, \tag{2}$$

where λ^{-1} is the mean time between switches, that is, $\lambda/2$ is the "average frequency" of the random square wave $F(t)$. The thermal fluctuations in Eq. (1) are given by $\xi(t)$, which is zero-mean Gaussian white noise with correlation function given by

$$\langle \xi(t)\xi(t')\rangle = k_B T\delta(t-t'). \tag{3}$$

We also assume that the external random signal and the internal random noise are independent processes.

Let $S(x,t)$ be the probability that at time t the particle is still in an interval (z_1,z_2) where z_1 and z_2 are absorbing boundaries. In the absence of the external random wave $F(t)$ the survival probability $S(x,t)$ of process (1) obeys the equation [12]

$$\frac{\partial S(x,t)}{\partial t}=f(x)\frac{\partial S(x,t)}{\partial x}+\frac{D}{2}\frac{\partial^2 S(x,t)}{\partial x^2}, \tag{4}$$

where $D=k_BT$. The initial condition is given by

$$S(x,0)=\begin{cases}1 & \text{if } z_1<x<z_2\\ 0 & \text{elsewhere}\end{cases} \tag{5}$$

and the boundary conditions are

$$S(z_1,t)=S(z_2,t)=0. \tag{6}$$

Let us now obtain the equation for the survival probability of process (1) when $F(t)\neq 0$. To this end it is convenient to decompose this function into two components,

$$S(x,t)=\tfrac{1}{2}[S^+(x,t)+S^-(x,t)], \tag{7}$$

where, for example, $S^+(x,t)$ is the survival probability given that $F(t)$ is in the state $+a$. An analogous definition applies for the minus state. As is well known, the functions $S^\pm(x,t)$ obey the following set of coupled partial differential equations [8,13]:

$$\mathscr{L}S^+ = +a\frac{\partial S^+}{\partial x} + \lambda(S^- - S^+), \tag{8}$$

$$\mathscr{L}S^- = -a\frac{\partial S^-}{\partial x} + \lambda(S^+ - S^-), \tag{9}$$

where $\mathscr{L}$ is the differential operator

$$\mathscr{L} \equiv \frac{\partial}{\partial t} - f(x)\frac{\partial}{\partial x} - \frac{D}{2}\frac{\partial^2}{\partial x^2}. \tag{10}$$

The initial conditions for (8) and (9) are

$$S^\pm(x,0) = 1 \quad (z_1 < x < z_2), \tag{11}$$

and the boundary conditions read

$$S^\pm(z_1,t) = S^\pm(z_2,t) = 0. \tag{12}$$

The equation for the total survival probability given by Eq. (7) is obtained by combining Eqs. (8) and (9); the final result is the following fourth-order partial differential equation (see Appendix A for details):

$$\mathscr{L}^2 S + [2\lambda - f'(x)]\mathscr{L}S - a^2\frac{\partial^2 S}{\partial x^2} = 0, \tag{13}$$

where $\mathscr{L}^2 S \equiv \mathscr{L}\mathscr{L}S$. This equation has to be solved under two initial conditions,

$$S(x,0) = 1, \quad \left.\frac{\partial S}{\partial t}\right|_{t=0} = 0, \tag{14}$$

where $z_1 < x < z_2$, and four boundary conditions,

$$S(z_i,t) = 0, \tag{15}$$

and

$$\left[\frac{D}{2}\mathscr{L} - \frac{D}{2}\frac{\partial}{\partial x}f(x) - f^2(x) + a^2\right]\left.\frac{\partial S}{\partial x}\right|_{x=z_i} = 0, \tag{16}$$

where $i = 1,2$ (cf. Appendix A).

Let us now obtain the equation satisfied by the first-passage-time moments, which in terms of the survival probability $S(x,t)$ are defined by [12]

$$T_n(x) = n\int_0^\infty t^{n-1}S(x,t)\,dt, \tag{17}$$

$n = 1,2,3,\ldots$, and $T_0(x) \equiv 1$. The combination of this equation with Eq. (13) followed by some integration by parts and the use of Eq. (14) yields

$$\mathscr{L}_D^2 T_n(x) - [2\lambda - f'(x)]\mathscr{L}_D T_n(x) - a^2\frac{\partial^2 T_n(x)}{\partial x^2} = G_n(x), \tag{18}$$

$(n = 1,2,3,\ldots)$, where $\mathscr{L}_D$ is the Fokker-Planck operator:

$$\mathscr{L}_D = f(x)\frac{\partial}{\partial x} + \frac{D}{2}\frac{\partial^2}{\partial x^2}, \tag{19}$$

and

$$G_n(x) = n[2\lambda - f'(x) - 2\mathscr{L}_D]T_{n-1}(x) - n(n-1)T_{n-2}(x). \tag{20}$$

The boundary conditions attached to Eq. (18) are [cf. Eqs. (15) and (16)]

$$T_n(z_i) = 0, \tag{21}$$

and

$$\left[\frac{D}{2}\mathscr{L}_D + \frac{D}{2}\frac{\partial}{\partial x}f(x) + f^2(x) - a^2\right]T_n'(x)\Big|_{x=z_i} = -\frac{nD}{2}T_{n-1}'(z_i), \tag{22}$$

where the prime denotes the derivative with respect to x. Equations (13) and (18) seem to be very difficult to solve, even in an approximate way, for a general deterministic force $f(X)$. Nevertheless, it is possible to obtain exact expressions of the first-passage-time moments in some cases. One of these is the case of the unbound particle for which $f(X) = 0$. We can find such a case in electrophoretic experiments with a random electrical field that switches between two constant values at random times governed by the probability density (2) [14].

Other exactly solvable models include that of a linear potential although in this case the amount of algebra involved notably increases. Therefore, and as an illustration of the formalism, we will only treat here the unbound process.

When $f(X) = 0$ the operator defined by Eq. (19) reduces to $\mathscr{L}_D = (D/2)d^2/dx^2$ and Eq. (18) for the first-passage-time moments reads

$$\frac{D^2}{4}\frac{d^4 T_n(x)}{dx^4} - (a^2 + \lambda D)\frac{d^2 T_n(x)}{dx^2} = g_n(x), \tag{23}$$

where

$$g_n(x) = 2n\lambda T_{n-1}(x) - nDT_{n-1}''(x) - n(n-1)T_{n-2}(x). \tag{24}$$

The four boundary conditions attached to Eq. (23) are given by Eq. (21) and by

$$\frac{D^2}{4}T_n'''(z_i) - a^2 T_n'(z_i) = -\frac{nD}{2}T_{n-1}'(z_i). \tag{25}$$

Equation (23) is a linear ordinary differential equation whose solution can be readily obtained. When $n = 1$ and $z_1 = 0$, $z_2 = L$ the solution of problems (23), (21), and (25) is [15]

$$T_1(x) = \frac{x(L-x)}{D+D'} + \frac{L}{a}\left(\frac{D'}{D+D'}\right)^{3/2}\frac{\sinh\mu x\,\sinh\mu(L-x)}{\sinh\mu L}, \tag{26}$$

where $D' \equiv a^2/\lambda$ is the intensity of the random telegraph signal, and

$$\mu \equiv \frac{a}{D}\left(\frac{D+D'}{D'}\right)^{1/2}. \tag{27}$$

It is worthwhile mentioning that Eq. (23) for the MFPT was solved for a trap at $z_1=0$ and a reflecting boundary condition at $z_2=L$ in [5].

We now discuss some asymptotic properties of $T_1(x)$. In this discussion it is more convenient to work in dimensionless units. If we assume that the white noise intensity D and the length of the interval L are finite, then the scaling will be given by the characteristic time L^2/D and the characteristic length L. This scaling is equivalent to setting $D=1$ and $L=1$ in Eq. (26), that is,

$$T_1(x)=\frac{\lambda}{\lambda+a^2}x(1-x)+\frac{a^2}{(\lambda+a^2)^{3/2}} \times\frac{\sinh[(\lambda+a^2)^{1/2}x]\sinh[(\lambda+a^2)^{1/2}(1-x)]}{\sinh[(\lambda+a^2)^{1/2}]}. \tag{28}$$

We distinguish two cases. (1) Suppose that $a^2/\lambda \ll 1$ with $\lambda \neq 0$. We have from Eq. (28) that $T_1(x)\approx x(1-x)$, which corresponds to the MFPT of a free first-order process driven by Gaussian white noise. This has to be certainly the case because in this situation the dichotomous signal is much weaker than diffusion and the MFPT corresponds to that of the process $\dot{X}(t)=\xi(t)$.

(2) Assume that $a^2/\lambda \gg 1$. We have from (28) that

$$T_1(x)\approx\frac{\lambda}{a^2}x(1-x)+\frac{1}{a}\frac{\sinh ax \sinh a(1-x)}{\sinh a}. \tag{29}$$

In this case, and roughly speaking, diffusion is weaker than the dichotomous signal but, in order to obtain asymptotic expressions for $T_1(x)$, we must distinguish two situations according to the order of magnitude of a/λ, the average distance traveled between switches of the dichotomous signal. (a) $a/\lambda \ll 1$. In this situation the first term on the right hand side of Eq. (29) dominates over the second term. Hence,

$$T_1(x)\approx\frac{\lambda}{a^2}x(1-x). \tag{30}$$

This corresponds to the Gaussian white noise limit of the dichotomous noise. This is not surprising since, as is well known, when the average distance traveled between switches is much less than the length of the interval the dichotomous noise acts as Gaussian white noise with a noise intensity given by a^2/λ. (b) Suppose now that $a/\lambda \gg 1$. In this case the second term on the right hand side of Eq. (29) dominates over the first and,

$$T_1(x)\approx\frac{1}{a}\frac{\sinh ax \sinh a(1-x)}{\sinh a}. \tag{31}$$

Note that this expression can be written in the form

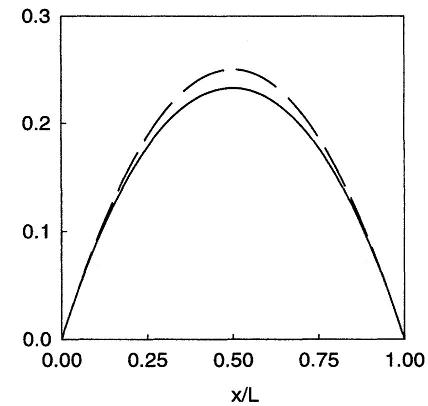

FIG. 1. MFPT as a function of the initial position. Parameter values: $a=D/L$, $\lambda=D/L^2$. The solid line corresponds to the combination of random signals. The dashed line corresponds to white noise alone.

$$T_1(x)\approx\tfrac{1}{2}[T^w_{+a}(x)+T^w_{-a}(x)],$$

where

$$T^w_{\pm a}(x)=\mp\frac{x}{a}\pm\frac{1}{a}\frac{1-e^{\mp 2ax}}{1-e^{\mp 2a}},$$

is the MFPT of the system $\dot{X}=\pm a+\xi(t)$. This is not a surprising result because when the average distance traveled between switches is much greater than the length of the interval, the dichotomous noise maintains (with probability 1) its initial value $F(0)=+a$ or $F(0)=-a$, which are assumed equally likely.

In Fig. 1 we plot the MFPT $T_1(x)$ as a function of the initial position x. The solid line corresponds to $T_1(x)$ for process (1) with $f(X)=0$. The dashed line corresponds to the MFPT in the absence of dichotomous noise, that is, it represents the MFPT for an unbound process driven by Gaussian white noise. It is interesting to note that the addition of an external dichotomous noise reduces the escape time of the system. This effect can be understood in the sense that the addition of the external random force produces an increment of the intensity of the total driving noise, $\xi(t)+F(t)$, of the free process.

In Fig. 2 we plot the MFPT at $x=L/2$ as a function of the "average random frequency" λ of the random square wave signal. We observe that T_1 is a monotonous function of λ and there is no sign of stochastic resonance behavior in the MFPT. This confirms some results obtained by means of numerical simulations of the discrete model [16]. Note that $T_1(0.5)=0.05$ as $\lambda\to 0$, in agreement with Eq. (31), and that $T_1(0.5)$ goes to 0.25 when $\lambda\to\infty$, which corresponds to the Gaussian white noise limit.

Note that we have discussed the limit of $T_1(x)$ in the absence of external random force. The opposite case consists in taking D equal to zero. When $D=0$ we have from Eq. (26) that

$$T_1(x)\approx\frac{L}{2a}+\frac{\lambda}{a^2}(L-x)x, \tag{32}$$

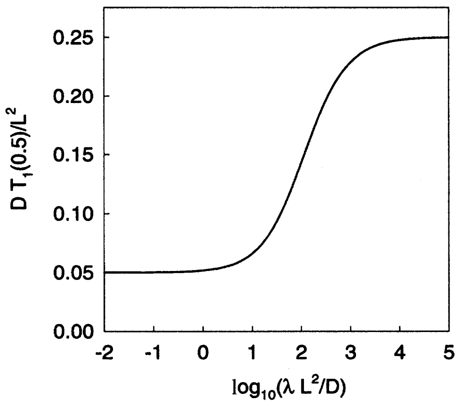

FIG. 2. $T_1(L/2)$ as a function of λ. Parameter values: $a=10D/L$.

provided that $x\neq 0, L$. This equation coincides with the expression of the MFPT for a free process driven by dichotomous Markov noise [17]. At the boundaries, $x=0$, L, the expression for $T_1(x)$ given by Eq. (26) does not converge to the value given by Eq. (32). This means that the limits $D\rightarrow 0$ and $x\rightarrow 0$, L do not commute, as has been previously noted [8]. The reason for this can be found in the fact that the boundary condition (21) is only valid if $D\neq 0$. Figure 3 shows the convergence of $T_1(x)$ to expression (32) when $D\rightarrow 0$ and illustrates the singular behavior at the boundaries.

Following an analogous calculation one can obtain higher-order moments of the first passage time. Thus, when $n=2$ the solution of the problem given by Eqs. (23), (21), and (25) reads

$$\begin{aligned} T_2(x) = {} & A_1 x^2 (L-x)^2 + A_2 x(L-x) \\ & + A_3[(2x-L)\sinh \mu(2x-L) - L \sinh \mu L] \\ & + A_4 \frac{\sinh \mu x \sinh \mu(L-x)}{\sinh \mu L}, \end{aligned} \tag{33}$$

where μ is given by Eq. (27) and

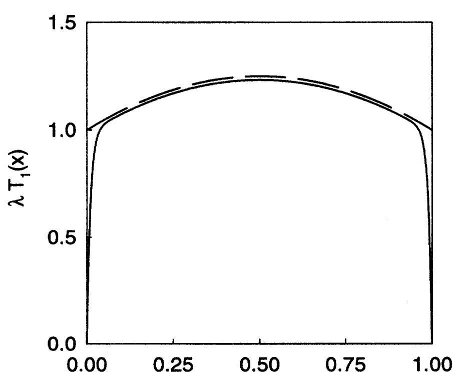

FIG. 3. $T_1(x)$ as a function of the initial position (solid line) for $D=0.04\lambda L^2$, $a=0.5L\lambda$. Dashed line shows the MFPT for a free system driven by dichotomous Markov noise ($a=0.5L\lambda$).

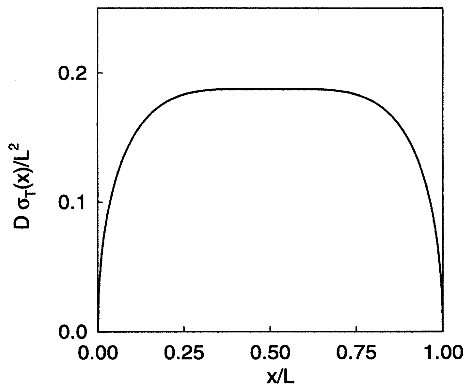

FIG. 4. Standard deviation, $\sigma_T(x)$, of the first passage time as a function of the initial position. Parameter values: $a=D/L$, $\lambda=D/L^2$.

$$A_1 = \frac{1}{3(D+D')^2}, \tag{34}$$

$$A_2 = \frac{L^2}{3(D+D')^2} - \frac{1}{a^2}\left(\frac{D'}{D+D'}\right)^3 + \frac{L}{aD'}\left(\frac{D'}{D+D'}\right)^{5/2} \coth \mu L, \tag{35}$$

$$A_3 = \frac{D'^2 L(2D'+D)}{4a^2(D+D')^3 \sinh \mu L}, \tag{36}$$

and

$$A_4 = \frac{L^3}{3aD'}\left(\frac{D'}{D+D'}\right)^{5/2} + \frac{DL(2D'-D)}{2a^3 D'}\left(\frac{D'}{D+D'}\right)^{7/2} + \frac{L^2(4D'+D)}{2a^2 D'}\left(\frac{D'}{D+D'}\right)^3 \coth \mu L. \tag{37}$$

In Fig. 4 we plot the standard deviation $\sigma_T(x)$ of the first passage time as a function of x, i.e.,

$$\sigma_T(x) = \sqrt{T_2(x) - T_1^2(x)}.$$

We note that $\sigma_T(x)$ can be considered constant except in the regions near the boundaries; this means that the dispersion of the first passage time around its mean value is almost constant except near the boundaries.

The behavior of $\sigma_T(L/2)$ as a function of λ is completely analogous to that of Fig. 2 for the MFPT and it shows that there is no resonant effect for $\sigma_T(x)$ either. Nevertheless, the dispersion presents an interesting behavior when we consider $\sigma_T(x)$ as a function of the "total noise intensity" defined by the sum $D+D'=D+a^2/\lambda$. Thus, plotting $\sigma_T(L/2)$ as a function of D with the prescription that the total noise intensity remains const, $D+a^2/\lambda=\text{const}$, we observe that the dispersion is not monotonic and an intermediate maximum appears (see Fig. 5). Note that the values $D=0$ and $D=10$ correspond to a system that is driven exclusively either by dichotomous noise ($D=0$) or by Gaussian white noise

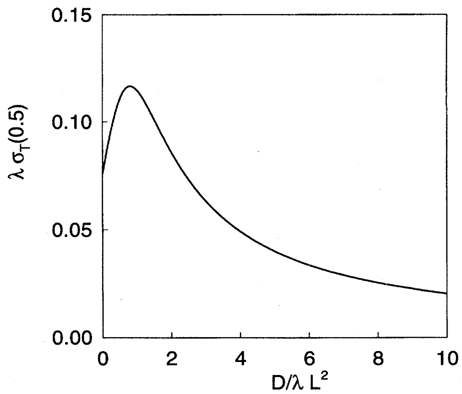

FIG. 5. $\sigma_T(L/2)$ as a function of D provided that the total noise intensity $D+a^2/\lambda$ is constant. Parameter values: $a^2=L^2\lambda^2\sqrt{10-D/\lambda L^2}$.

$(D=10)$. As the two driving noises have different properties, the nonmonotonic behavior arises from this mixture. A possible reason for this behavior may be that the addition of a small amount of noise of different nature (around either $D=0$ or $D=10$) does not change substantially the MFPT but increases dispersion of FPT. Therefore, it originates in both cases an increment of the standard deviation that leads in the maximum value shown in Fig. 5.

We finally note that starting from Eq. (23) and following in an analogous way we can obtain higher-order moments. However, in the case of an unbound particle, we can get a closed expression for the Laplace transform of the survival probability,

$$\hat{S}(x,s)=\int_0^\infty e^{-st}S(x,t)dt,$$

from which all first-passage-time moments can be obtained through the relation

$$T_n(x)=(-1)^{n-1}\frac{\partial^{n-1}\hat{S}(x,s)}{\partial s^{n-1}}\bigg|_{s=0},$$

with $\hat{S}(x,0)=T_1(x)$. In Appendix B we present the complete expression of $\hat{S}(x,s)$ for the unbound process.

We briefly summarize the main results achieved. We have studied the first-passage-time statistics for a dynamical system that is a combination of a diffusion process and an external random force. We have shown that (for the unbound process) the addition of a random dichotomous noise reduces the escape time of the system out of a finite interval. Nevertheless, this does not mean a resonant behavior of the system since there is no coherent motion [3] associated with any characteristic time scale of the system (such as the mean time between switches of the external random force). We have also obtained exact expressions of the second moment of the first passage time for the unbound particle. This calculation shows a nonmonotonic behavior of the variance when $\sigma_T(x)$ is considered a function of the total noise intensity.

This work has been supported in part by Dirección General de Investigación Científica y Técnica under Contract No. PB93-0812, and by Societat Catalana de Física (Institut d'Estudis Catalans).

APPENDIX A: EQUATION FOR THE SURVIVAL PROBABILITY

From Eqs. (8) and (9) we get

$$\mathscr{L}S=a\frac{\partial\Delta}{\partial x},\tag{A1}$$

$$\mathscr{L}\Delta=a\frac{\partial S}{\partial x}-2\lambda\Delta,\tag{A2}$$

where S is the total survival probability given by Eq. (7), and $\Delta=(S^+-S^-)/2$. The derivative with respect to x of Eq. (A2) and the use of Eq. (A1) yield

$$\frac{\partial}{\partial x}\mathscr{L}\Delta=a\frac{\partial^2 S}{\partial x^2}-\frac{2\lambda}{a}\mathscr{L}S,\tag{A3}$$

but from the definition of the operator $\mathscr{L}$ we have

$$\frac{\partial}{\partial x}\mathscr{L}\Delta=\frac{1}{a}[\mathscr{L}^2S-f'(x)\mathscr{L}S].$$

The substitution of this equation into Eq. (40) results in Eq. (13).

From Eqs. (7) and (11) we see that one initial condition for Eq. (13) is given by $S(x,0)=1$. On the other hand $\Delta(x,0)=0$ and from Eqs. (10) and (A1) we have

$$\frac{\partial S}{\partial t}\bigg|_{t=0}=S\,|_{t=0}=0,$$

which agrees with Eq. (14).

Let us now obtain the boundary conditions given by Eqs. (15) and (16). From Eqs. (10) and (A1) we have

$$\mathscr{L}\Delta=\frac{\partial\Delta}{\partial t}-\frac{f(x)}{a}\mathscr{L}S-\frac{D}{2a}\frac{\partial}{\partial x}(\mathscr{L}S).$$

Substituting this into Eq. (A2) and taking into account

$$S(z_i,t)=\Delta(z_i,t)=0,\tag{A4}$$

we get

$$\frac{f(x)}{a}\mathscr{L}S|_{x=z_i}+\frac{D}{2a}\frac{\partial}{\partial x}(\mathscr{L}S)\bigg|_{x=z_i}=-a\frac{\partial S}{\partial x}\bigg|_{x=z_i},\tag{A5}$$

but [cf. Eqs. (10) and (A4)]

$$\mathscr{L}S|_{x=z_i}=-\left[f(x)\frac{\partial S}{\partial x}+\frac{D}{2}\frac{\partial^2 S}{\partial x^2}\right]\bigg|_{x=z_i},$$

and

$$\frac{\partial}{\partial x}(\mathscr{L}S)\bigg|_{x=z_i}=\left[\frac{\partial^2 S}{\partial x\partial t}-f'(x)\frac{\partial S}{\partial x}-f(x)\frac{\partial^2 S}{\partial x^2}-\frac{D}{2}\frac{\partial^3 S}{\partial x^3}\right]\bigg|_{x=z_i}.$$

The substitution of these two equations into Eq. (A2) and some reorganization of terms finally yield Eq. (16).

APPENDIX B: SURVIVAL PROBABILITY FOR THE UNBOUND PROCESS

When $f(X)=0$ we see from Eqs. (13)–(16) that $\hat{S}(x,s)$ satisfies the following differential equation:

$$\frac{D^2}{4}\frac{\partial^4\hat{S}}{\partial x^4}-[D(\lambda+s)+a^2]\frac{\partial^2\hat{S}}{\partial x^2}+s(2\lambda+s)\hat{S}=s+2\lambda, \tag{B1}$$

with boundary conditions

$$\hat{S}(0,s)=\hat{S}(L,s)=0, \tag{B2}$$

$$\left[\frac{D^2}{4}\frac{\partial^3\hat{S}}{\partial x^3}-(a^2+Ds/2)\frac{\partial\hat{S}}{\partial x}\right]\bigg|_{x=z_i}=0, \tag{B3}$$

where $z_i=0$, L. The solution to this problem is

$$\hat{S}(x,s)=\frac{1}{s}\left[\frac{\cosh\alpha_-L/2-\cosh\alpha_-(x-L/2)}{\cosh\alpha_-L/2}+\tanh\alpha_-L/2\frac{\cosh\alpha_-(x-L/2)\cosh\alpha_+L/2-\cosh\alpha_+(x-L/2)\cosh\alpha_-L/2}{\sinh\alpha_-L/2\ \cosh\alpha_+L/2-A\ \sinh\alpha_+L/2\ \cosh\alpha_-L/2}\right], \tag{B4}$$

where

$$\alpha_\pm^2=\frac{2}{D^2}[D(\lambda+s)+a^2\pm\sqrt{(D\lambda+a^2)^2+2a^2Ds}], \tag{B5}$$

$$A=\frac{\alpha_+(\alpha_+^2-\beta)}{\alpha_-(\alpha_-^2-\beta)}, \tag{B6}$$

and

$$\beta=\frac{4}{D^2}(a^2+Ds/2). \tag{B7}$$

[1] F. Moss, in *Contemporary Problems in Statistical Physics*, edited by G. H. Weiss (SIAM, Philadelphia, 1994), and references therein; K. Weisenfeld and F. Moss, Nature **373**, 33 (1995).
[2] J. E. Fletcher, S. Havlin, and G. H. Weiss, J. Stat. Phys. **51**, 215 (1989).
[3] J. Masoliver, A. Robinson, and G. H. Weiss, Phys. Rev. E **51**, 4021 (1995).
[4] A. R. Bulsara, S. B. Lowen, and C. D. Rees, Phys. Rev. E **49**, 4989 (1994).
[5] C. R. Doering and J. C. Godoua, Phys. Rev. Lett. **69**, 2318 (1992).
[6] U. Zücher and C. R. Doering, Phys. Rev. E **47**, 3862 (1993).
[7] M. Bier and R. D. Astumian, Phys. Rev. Lett. **71**, 1649 (1993).
[8] V. Balakrishnan, C. Van den Broeck, and P. Hanggi, Phys. Rev. A **38**, 4213 (1988).
[9] C. R. Doering, W. Horthemke, and J. Riordan, Phys. Rev. Lett. **72**, 2984 (1994).
[10] J. Masoliver and G. H. Weiss, Physica A **149**, 395 (1988).
[11] P. J. Brockwell, Adv. Appl. Prob. **17**, 42 (1985).
[12] G. H. Weiss, Adv. Chem. Phys. **13**, 1 (1966).
[13] N. Van Kampen, Physica A **96**, 435 (1979).
[14] M. Bier, A. Palunski, R. A. Mosher, and D. A. Saville, Science **219**, 1281 (1983); A. Aldroubi and M. M. Garner, BioTechniques **13**, 620 (1992); J. Masoliver and G. H. Weiss, Sep. Sci. Tech. **31**, 319 (1996).
[15] Equation (26) could also be obtained from Eq. (3.31) of Ref. [8] after lengthy calculations.
[16] G. H. Weiss (private communication).
[17] J. Masoliver, K. Lindenberg, and B. J. West, Phys. Rev. A **33**, 2177 (1986); **34**, 1481 (1986).